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Single Cell Discretizations of Order Two and Four for Biharmonic Problems

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New difference formulas are derived for solving the biharmonic problem in two dimensions over a rectangular domain. These methods use only the nine grid points of a single mesh cell and do not require fictitious points in order to approximate the boundary conditions. Derivatives of the solution are obtained as a by-product of the methods. Second order formulas are derived for both the first and second biharmonic problems. In numerical experiments, the new second order formulas compare favourably with the standard second order methods. Extensions to fourth order formulas are given. The method of deriving these formulas can be used to derive similar formulas for arbitrarily shaped regions.

1. INTRODUCTION

We consider the biharmonic equation

$$Lu = u_{xxxx} + 2u_{xxyy} + u_{yyyy} = f(x, y) \tag{1}$$

with two types of boundary conditions. In the first case we consider the boundary conditions

$$u, \partial u/\partial n \text{ prescribed} \tag{2}$$

which we call the first problem. In the second case we consider the boundary conditions

$$u, \partial^2 u/\partial n^2 \text{ prescribed} \tag{3}$$

which we will refer to as the second problem.

We restrict our attention to regions which may be partitioned into square subregions by a uniform grid. The finite difference approximation to Eq. (1) is obtained on a square subregion that consists of the central point 0, (x_0, y_0) and the eight points $(x_0 \pm h, y_0)$, $(x_0, y_0 \pm h)$, $(x_0 \pm h, y_0 \pm h)$ denoted by $1 - 8$ (see Fig. 1). A combination of the values of the solution $u(x, y)$, its derivatives and the values of the right hand function $f(x, y)$ at the nine grid points are used to derive approximation

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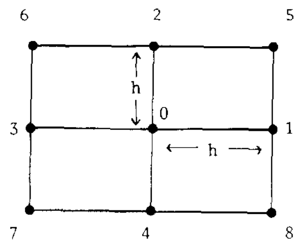


FIGURE 1

formulas. We denote the values of u, u_x, u_y, u_{xx}, u_{yy} and f at the grid point j by u_j, u_{xj}, u_{yj}, u_{xxj}, u_{yyj} and f_j, respectively.

The standard 13-point approximation of the biharmonic equation is obtained by using second order central differences. This requires the use of fictitious points outside of the region of interest. The values of the solution at these points is expressed in terms of the boundary values of the derivative and the values of the solution at grid points inside the region. Gupta and Manohar [1] have considered several such schemes for the first problem and have shown that the accuracy of the numerical solution depends upon the boundary approximation used.

The finite difference formulas which we present here are based on only the eight grid points surrounding each grid point. This means that fictitious points for incorporating the boundary conditions are not required. However, the linear systems that are generated by these formulas have a more complicated structure than those derived from the standard central-difference formulas, and they are not positive-definite. Consequently, the linear systems have to be solved by direct methods in most cases. Successive over relaxation (S.O.R.) was used also, but convergence was slow.

In spite of these drawbacks, it is hoped that the new ideas presented in this paper may lead to the development of new techniques for solving more general problems.

2. Embedded Polynomials

The procedure we use to derive the finite difference formulas is new, although it has connections with the "Mehrstellenverfahren" of Collatz [2] and the works of Young and Dauwalder [3] and that of Lynch and Rice [4], and Boisvert [5]. Our experience with the finite element method [6] also helped us formulate this new approach. The development of the procedure for other linear differential equations has been given by Gupta, Manohar and Stephenson in [7–11].

For the derivation of a difference formula for a grid cell shown in Fig. 1, we require that the solution u and the function f can be expanded in a power series. Let us assume the following expansions

$$u=\sum a_{i,j}x^iy^j, \qquad f=\sum f_{i,j}x^iy^j \tag{4}$$

where x and y are local coordinates with origin at the center 0 of the grid element. If we substitute expansions (4) in the differential equation (1) and equate coefficients of each monomial $x^i y^j$ we get the linear constraints on the unknown coefficients $a_{i,j}$:

$$(i+4)(i+3)(i+2)(i+1)\,a_{i+4,j} + 2(i+2)(i+1)(j+2)(j+1)\,a_{i+2,j+2} + (j+4)(j+3)(j+2)(j+1)\,a_{i,j+4} = f_{i,j}, \qquad i,j = 0, 1, 2,.... \tag{5}$$

The coefficients $f_{i,j}$ are assumed to be known, or can be expressed in terms of the grid point values of $f(x, y)$.

In order to find a finite difference discretization for (1) we first of all truncate the series for u in (4) so that the solution for the boundary value problem is represented by a polynomial on each grid cell. This embedded polynomial is uniquely determined from a subset of Eqs. (5) and the values of the polynomial and its derivatives at the grid points on the single cell. These values, which are eventually found as the solution of the discretized system of equations, are the values of the numerical solution of the boundary value problem and its derivatives. The interpolation equations which relate the coefficients $a_{i,j}$ of the embedded polynomial to the values of the solution u and its derivatives at the grid points are

$$\begin{aligned}
u_k &= u(x_k, y_k) = \sum a_{i,j} x_k^i y_k^j \\
u_{xk} &= \frac{\partial u}{\partial x}(x_k, y_k) = \sum i a_{i,j} x_k^{i-1} y_k^j \\
u_{yk} &= \frac{\partial u}{\partial y}(x_k, y_k) = \sum j a_{i,j} x_k^i y_k^{j-2} \\
u_{xxk} &= \frac{\partial^2 u}{\partial x^2}(x_k, y_k) = \sum i(i-1)\,a_{i,j} x_k^{i-2} y_k^j \\
u_{yyk} &= \frac{\partial^2 u}{\partial y^2}(x_k, y_k = \sum j(j-1)\,a_{i,j} x_k^i y_k^{j-2}.
\end{aligned} \tag{6}$$

In the case of the first biharmonic problem we use only the values of u_k, u_{xk} and u_{yk}. In the case of the second biharmonic problem we use the values of u_k, u_{xxk} and u_{yyk}. We form a system of linear equations from (5) and (6) which is solved to determine the coefficients $a_{i,j}$ of the embedded polynomial. These coefficients are expressed in terms of the values of u and its derivatives at grid points in the grid cell, and the coefficients of f. The finite difference equations for the values of the solution and its derivatives at the center of a grid cell are given by

$$\begin{aligned}
u_0 &= u(0, 0) = a_{0,0} \\
u_{x0} &= a_{1,0}, \qquad u_{y0} = a_{0,1} \\
u_{xx0} &= 2a_{2,0}, \qquad u_{yy0} = 2a_{0,2}.
\end{aligned}$$

Inversion of the system of linear equations (5) and (6) is easily done with a computer and is essential in the case when arbitrarily shaped regions are considered. However, on a square grid cell, we are able to take advantage of certain symmetries, as will be illustrated in the next section. In fact, finite difference formulas can be found without inverting the system of linear equations. For instance, in the case of the first biharmonic problem, we only make use of u, u_x and u_y at the grid points. Consequently, we only need to determine the first three coefficients $a_{0,0}$, $a_{1,0}$ and $a_{0,1}$ for the required difference equations.

The interpolations of u and its derivatives are not necessarily done on the same set of grid points, and care must be taken to avoid a singular system of equations. For example, if u is represented by a polynomial of degree n, there should be no more than $n + 1$ interpolation conditions on any grid line in the computational cell.

In the following discussion we attempt to justify the choices we made in selecting the degree of the embedded polynomials, and the interpolation conditions (6) that we consider in this paper.

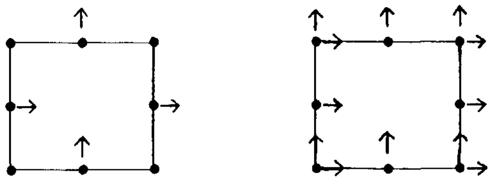
We note that in the expansion of $u(x, y)$, terms of degree $m > 3$ are partitioned into two sets by the constraints (5). For instance, the coefficients $a_{6,0}$, $a_{4,2}$, $a_{2,4}$ and $a_{0,6}$ are associated with $f_{2,0}$ and $f_{0,2}$ while $a_{5,1}$, $a_{3,3}$ and $a_{1,5}$ are associated with $f_{1,1}$. We adopt the convention of grouping the interelated terms of degree M together into two sets, the first set containing the terms connected with the term $a_{M,0}x^M$ and the second set containing the term connected with the term $a_{M-1,1}x^{M-1}y$. We refer to the polynomial expansion that includes just the terms up to the first set of terms of degree M as being an expansion up to degree $M - \frac{1}{2}$. In Table I we list the number of interpolations required to determine the coefficients $a_{i,j}$ for expansions of u up to degree 6.

In Fig. 2 we illustrate two natural choices of interpolations. We indicate by $\bullet$, a grid point where the value of u is interpolated, and by $\rightarrow$, a grid point where a derivative of u is interpolated.

The number of interpolation conditions in Figs. 2a and b is 12 and 20, respectively. Consequently, the degree of the embedded polynomials chosen in the sequel is 3.5 and 5.5, respectively. With these choices, the system of linear equations (5) and (6) are nonsingular and the resulting difference schemes can be determined by a computer routine.

TABLE I

Degree of polynomial u	Number of terms in u	Number of constraints	Number of interpolations
3.5	13	1	12
4	15	1	14
4.5	18	2	16
5	21	3	18
5.5	25	5	20
6	28	6	22



2a 2b

FIGURE 2

3. FORMULAS FOR THE FIRST BIHARMONIC PROBLEM

3.1. *Second Order Formula*

In this section we derive a new second order formula which has two advantages over the classical thirteen point formula. It is based upon a single computational cell, and incorporates the boundary conditions in a natural way without the need to introduce fictitious points or special schemes at the boundary. We give the details of the derivation in this case in order to illustrate the general procedure which was outlined previously in Section 2.

A second order formula can be found by considering a polynomial expansion for u of degree 3.5. We interpolate u at the points 1, 2,..., 8, u_x at the points 1, 3 and u_y at the points 2, 4 in Fig. 1. These 12 conditions together with one constraint from (5) determine the 13 coefficients in the polynomial u.

We let $u = a_{0,0} + a_{1,0}x + a_{0,1}y + \cdots a_{4,0}x^4 + a_{2,2}x^2y^2 + a_{0,4}y^4$, the first constraint $(i = 0, j = 0)$ from (5) multiplied by h^4 is

$$f_{0,0}h^4 = (24a_{4,0} + 8a_{2,2} + 24a_{0,4})\,h^4. \tag{7.1}$$

The interpolation conditions give

$$\begin{aligned}
u_1 &= a_{0,0} + a_{1,0}h + a_{2,0}h^2 + a_{3,0}h^3 + a_{4,0}h^4\\
u_2 &= a_{0,0} + a_{0,1}h + a_{0,2}h^2 + a_{0,3}h^3 + a_{0,4}h^4\\
&\;\vdots\\
u_8 &= a_{0,0} + a_{1,0}h - a_{0,1}h + a_{2,0}h^2 - a_{1,1}h^2 + a_{0,2}h^2 + a_{3,0}h^3\\
&\quad - a_{2,1}h^3 + a_{1,2}h^3 - a_{0,3}h^3 + a_{4,0}h^4 + a_{2,2}h^4 + a_{0,4}h^4
\end{aligned}$$

and (7.2)

$$\begin{aligned}
u_{x1} &= a_{1,0} + 2a_{2,0}h + 3a_{3,0}h^2 + 4a_{4,0}h^3\\
&\;\vdots\\
u_{y4} &= a_{0,1} - 2a_{0,2}h + 3a_{0,3}h^2 - 4a_{0,4}h^3.
\end{aligned}$$

Inversion of this system (7.1), (7.2) by hand is not necessary. We are only interested in finding expressions for $a_{0,0}$, $a_{1,0}$ and $a_{0,1}$ since these give the required finite difference expressions for u_0, u_{x0} and u_{y0}, respectively. For a square grid element, we are able to make use of the following symmetric expression:

$$\begin{aligned}\Diamond u_0 &= u_1 + u_2 + u_3 + u_4 \\ &= 4a_{0,0} + 2(a_{2,0} + a_{0,2})\,h^2 + 2(a_{4,0} + a_{0,4})\,h^4 \qquad (8.1)\end{aligned}$$

$$\begin{aligned}\Box u_0 &= u_5 + u_6 + u_7 + u_8 \\ &= 4a_{0,0} + 4(a_{2,0} + a_{0,2})\,h^2 + 4(a_{4,0} + a_{2,2} + a_{0,4})\,h^4 \qquad (8.2)\end{aligned}$$

$$\begin{aligned}h\Diamond u_0' &= h(u_{x1} - u_{x3} + u_{y2} - u_{y4}) \\ &= 4(a_{2,0} + a_{0,2})\,h^2 + 8(a_{4,0} + a_{0,4})\,h^4. \qquad (8.3)\end{aligned}$$

Equations (7.1), (8.1), (8.2) and (8.3) can be considered as four linear equations in the four unknowns $a_{0,0}$, $(a_{2,0} + a_{0,2})\,h^2$, $a_{2,2}h^4$ and $(a_{4,0} + a_{0,4})\,h^4$. Elimination gives us $a_{0,0}$ and hence a finite difference scheme for u_0

$$u_0 = a_{0,0} = \frac{2}{7}\Diamond u_0 - \frac{1}{28}\Box u_0 - \frac{3h}{28}\Diamond u_0' + \frac{1}{56}f_{0,0}h^4. \qquad (9.1)$$

In a similar fashion, from

$$u_1 - u_3 = 2a_{1,0}h + 2a_{3,0}h^3$$

and

$$h(u_{x1} + u_{x3}) = 2a_{1,0}h + 6a_{3,0}h^3$$

we derive an expression for $a_{1,0}h$, which gives a difference scheme for hu_{x0},

$$hu_{x0} = \frac{3}{4}(u_1 - u_3) - \frac{h}{4}(u_{x1} + u_{x3}). \qquad (9.2)$$

Similarly, we get

$$hu_{y0} = \frac{3}{4}(u_2 - u_4) - \frac{h}{4}(u_{y2} + u_{y4}). \qquad (9.3)$$

Numerical results in Section 5 show that this scheme compares favourably with the classical thirteen point scheme. At interior mesh points we have three unknowns u, u_x and u_y. This means that the number of bands with nonzero entries is increased and so is the size of the final matrix for the same mesh size. However, the accuracy for a given mesh size is improved by this new scheme, and in addition, the values of the derivatives of the solution, which are often of interest, are also computed.

Because of symmetry in a square grid cell, formulas (9) are exact for polynomials of degree 5 even though the derivation was based upon polynomials of a lesser degree. If we had included polynomials of degree 5 in the derivation, the symmetric expressions (8) would not have altered. However, in the case when an irregular grid is used, we compute the difference formulas automatically by inverting the system (7) in an assembly subroutine in a manner similar to a finite element program. In this case, the formulas do not exhibit this superpotency, being exact only for polynomials of degree 3.5.

3.2. *Fourth Order Formulas*

A fourth order method can be found by considering a polynomial expansion for u up to degree 5.5. We interpolate u on the points 1–8, u_x on the points 1, 3, 5, 6, 7, 8 and u_y on the points 2, 4, 5, 6, 7, 8. These 20 conditions together with 5 constraints from (5) determine the 25 coefficients $a_{i,j}$ in the polynomial $u(x,y)$. We adopt the notation $\Box u_0' = u_{x5} - u_{x6} - u_{x7} + u_{x8} + u_{y5} + u_{y6} - u_{y7} - u_{y8}$. The difference formulas are

$$u_0 = \frac{3}{11} \Diamond u_0 - \frac{1}{44} \Box u_0 - \frac{7}{66} h \Diamond u_0' - \frac{1}{264} h \Box u_0'$$

$$+ \frac{5}{264} f_{0,0} h^4 + \frac{1}{396} (f_{2,0} + f_{0,2}) h^6 \tag{10.1}$$

$$hu_{x0} = \frac{6}{10} (u_1 - u_3) + \frac{3}{40} (u_5 - u_6 - u_7 + u_8) - \frac{2h}{10} (u_{x1} + u_{x3})$$

$$- \frac{h}{40} \Box u_{x0} - \frac{h}{40} (u_{y5} - u_{y6} + u_{y7} - u_{y8}) + \frac{1}{120} f_{1,0} h^5 \tag{10.2}$$

$$hu_{y0} = \frac{6}{10} (u_2 - u_4) + \frac{3}{40} (u_5 + u_6 - u_7 - u_8) - \frac{2h}{10} (u_{y2} + u_{y4})$$

$$- \frac{h}{40} \Box u_{y0} - \frac{h}{40} (u_{x5} - u_{x6} + u_{x7} - u_{x8}) + \frac{1}{120} f_{0,1} h^5. \tag{10.3}$$

For the computations reported in Section 5 we use interpolation of f in place of the Taylor coefficients. We make the following substitutions

$$f_0 = f_{0,0}, \qquad \frac{1}{2h} (f_1 - f_3) = f_{1,0}, \qquad \frac{1}{2h} (f_2 - f_4) = f_{0,1},$$

$$\frac{1}{2h^2} (f_1 - 2f_0 + f_3) = f_{2,0}, \qquad \frac{1}{2h^2} (f_2 - 2f_0 + f_4) = f_{0,2}.$$

Although the number of bands with nonzero entries is 27 which is greater than when the 13-point formula is used, the increased accuracy of formula (10) permits the use of a much coarser mesh. Because of symmetry, in a square grid element, formulas (10) are exact for polynomials of degree 7 even though they are based upon polynomials of a lesser degree. However, in the case when an irregular grid is used, the formulas do not exhibit this superpotency, being exact for polynomials of degree 5.5.

In the case when the value of u is known explicitly on the boundary and the boundary is parallel to one of the coordinates, then we also know the tangential derivative and formulas (10) can be used. However, in practice, u is only assumed to be known on the boundary nodes, and the tangential derivatives can only be determined by interpolation. We attempted to replace the derivatives along the boundary by the five point central difference formula

$$u' = \frac{1}{12k}(u_{-2} - 8u_{-1} + 8u_1 - u_2) + O(k^4). \tag{11}$$

Experiments showed us that the accuracy of formulas (10) were maintained only when the stepsize k was reduced to the order of 10^{-5}, a value much smaller than the mesh width $h = 1/16$.

We have developed an alternative procedure, which is applicable even in more general situations. We use special single cell formulas for grid cells adjacent to a boundary, or at a corner where two boundaries meet. These formulas use additional values of u at boundary points half way between the grid points on the boundary; see Fig. 3.

Because of the lack of symmetry, these formulas are more complicated than those presented earlier in (10). The formulas (15) and (16) for the two cases shown in Fig. 3 are given in the Appendix. The formulas for the other boundaries and corners are obtained from these by appropriate coordinate transformations. For instance, the left boundary formula is a reflection of the right boundary formula through the vertical line joining the vertices 2 and 4 with u_x replaced by $-u_x$. The upper boundary formula is obtained by reflecting the right boundary formula through the diagonal joining the vertices 7 and 5 with u_x and u_y interchanged. These formulas are not exact for all polynomials of degree 7, viz. x^6y and xy^6. However, numerical results in Section 5 show that these formulas maintain the fourth order accuracy of the scheme (10). Moreover, they can be generalized to non rectangular domains.

We illustrate a method of determining the orders of the formulas by considering

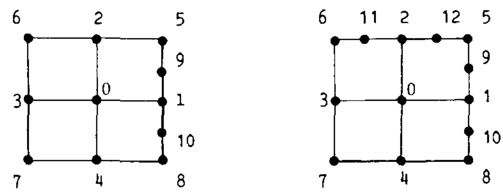


FIGURE 3

the formulas (9). We expand the values of u, u_x and u_y in Taylor series about the central point 0, and obtain the expressions

$$\Diamond u_0 = 4u_0 + \frac{2h^2}{2!}(u_{xx} + u_{yy})_0 + \frac{2h^4}{4!}(u_{xxxx} + u_{yyyy})_0 + O(h^6)$$

$$\Box u_0 = 4u_0 + \frac{4h^2}{2!}(u_{xx} + u_{yy})_0 + \frac{4h^4}{4!}(u_{xxxx} + 6u_{xxyy} + u_{yyyy})_0 + O(h^6)$$

$$h\Diamond u_0' = 2h^2(u_{xx} + u_{yy})_0 + \frac{2h^4}{3!}(u_{xxxx} + u_{yyyy})_0 + O(h^6).$$

From these expressions and the differential equation $f = Lu$, we get at the central point

$$\frac{16\Diamond u_0 - 2\Box u_0 - 6h\Diamond u_0' - 56u_0}{h^4} + f_{0,0} = [f - Lu|_0 + O(h^2).$$

Thus the finite difference formula (9.1) is a second order discretization of the biharmonic equation (1). Similarly, the formulas (9.2) and (9.3) are second order discretizations of the derivatives. Using this method, we can show that formulas (10), (15) and (16) are fourth order discretizations.

4. Formulas for the Second Biharmonic Problem

In this case of the second biharmonic problem we can replace the interpolations of u_x and u_y by interpolations of u_{xx} and u_{yy} repectively. We change the notation slightly using

$$\Diamond u_0'' = u_{xx1} + u_{xx3} + u_{yy2} + u_{yy4}$$

$$\Box u_{xx0} = \sum_{i=5}^{8} u_{xxi}$$

$$\Box u_{yy0} = \sum_{i=5}^{8} u_{yyi}$$

$$\Box u_0'' = \sum_{i=5}^{8} (u_{xxi} + u_{yyi}).$$

A second order formula for the second biharmonic problem is given by

$$u_0 = \frac{4}{11}\Diamond u_0 - \frac{5}{44}\Box u_0 - \frac{3}{44}h^2\Diamond u_0'' + \frac{5}{88}h^4 f_{0,0} \tag{12.1}$$

$$h^2 u_{xx0} = \frac{18}{55}(u_1 + u_3) - \frac{48}{55}(u_2 + u_4) + \frac{3}{11}\Box u_0 + \frac{7}{110} h^2(u_{xx1} + u_{xx3})$$
$$+ \frac{9}{55} h^2(u_{yy2} + u_{yy4}) - \frac{3}{22} h^4 f_{0,0} \tag{12.2}$$

$$h^2 u_{yy0} = -\frac{48}{55}(u_1 + u_3) + \frac{18}{55}(u_2 + u_4) + \frac{3}{11}\Box u_0 + \frac{9}{55} h^2(u_{xx1} + u_{xx3})$$
$$+ \frac{7}{110} h^2(u_{yy2} + u_{yy4}) - \frac{3}{22} h^4 f_{0,0}. \tag{12.3}$$

A fourth order formula is given by

$$u_0 = \frac{83}{241}\Diamond u_0 - \frac{91}{964}\Box u_0 - \frac{29}{482} h^2 \Diamond u_0'' - \frac{17}{1928} h^2 \Box u_0''$$
$$+ \frac{125}{1928} h^4 f_{0,0} + \frac{35}{2892} h^6 (f_{2,0} + f_{0,2}) \tag{13.1}$$

$$h^2 u_{xx0} = \frac{423}{1205}(u_1 + u_3) - \frac{1023}{1205}(u_2 + u_4) + \frac{60}{241}\Box u_0$$
$$+ \frac{3599}{60250} h^2(u_{xx1} + u_{xx3}) + \frac{951}{120500} h^2 \Box u_{xx0}$$
$$+ \frac{8901}{60250} h^2(u_{yy2} + u_{yy4}) + \frac{837}{60250} h^2 \Box u_{yy0}$$
$$- \frac{141}{964} h^4 f_{0,0} - \frac{417}{24100} h^6 f_{2,0} - \frac{658}{24100} h^6 f_{0,2}. \tag{13.2}$$

The formula for $h^2 u_{yy0}$ can be obtain from the formula for $h^2 u_{xx0}$ by interchanging the subscripts (1 and 2), (3 and 4), the derivatives (u_{xx} and u_{yy}) and the coefficients ($f_{2,0}$ and $f_{0,2}$).

As in the case of the first biharmonic problem, the formulas (12) and (13) are superpotent, being exact for polynomials of degree 5 and 7 respectively.

The fourth order formulas (13) require knowledge of the second tangential derivative. As in the case of the formulas (10), we introduce special formulas at the boundaries and corners, and the appropriate formulas were generated in a computer subroutine. We list the table of weights generated in the Appendix for the right hand edge formulas and the right upper corner formulas.

5. Numerical Examples

We consider a selection of problems that have been studied by Gupta and Manohar [1]. These problems have been studied by other authors and provide a useful basis for comparison. In each case we took the unit square $0 \leqslant x, y \leqslant 1$ as the region of integration and covered it with a uniform grid of width h. Gupta and Manohar [1] only considered the first biharmonic problem and their results are the best available in the literature. We considered both the first and second biharmonic problems. Our results compare favourably with those obtained by Gupta and Manohar.

Because our methods use three unknowns at each grid point, the number of unknowns for a certain mesh size is much greater than the number of unknowns for standard difference methods. For instance, for $h = 1/4$, $1/8$ and $1/16$, the number of unknowns are 27, 147 and 675, respectively. The second order methods have 13 unknowns in each equation while the fourth order methods have 27 unknowns in each equation. The bandwidth in the case of $h = 1/16$ is 99. We used both direct solvers and iterative methods. However, iterative methods were very slow, but because of the machine storage limitations, it was necessary to use them for $h = 1/16$ In fact, SOR failed to converge for $h = 1/16$ in less than 10,000 iterations, The computations were performed on a DEC–20 in single precision.

We considered the following problems:

PROBLEM 1.

$$u = x^2 + y^2 - xe^x \cos y, \qquad Lu = 0.$$

PROBLEM 2.

$$u = [(x - x^2)(y - y^2)]^2,$$
$$Lu = 8[3y^2(1-y)^2 + 3x^2(1-x)^2 + (6x^2 - 6x + 1)(6y^2 - 6y + 1)].$$

PROBLEM 3.

$$u = (1 - \cos 2\pi x)(1 - \cos 2\pi y),$$
$$Lu = (2\pi)^4 \, [4 \cos(2\pi x) \cos(2\pi y) - \cos(2\pi x) - \cos(2\pi y)].$$

We give the results of the first biharmonic problem in Table II, where we compare the maximum absolute errors in the approximate solution u for the second and fourth order formulas (9) and (10), respectively. We also compare the results obtained when the tangential derivatives in (10) are replaced by the five point formula (11), and finally, when the formulas (15) and (16) are employed at the boundary. We use $*$ to indicate that the results were affected by roundoff errors. The notation we use is $.6990(-3) = 0.699010^{-3}$, EXACT $\leqslant 1.0(-7)$.

From the results we observe that replacing the tangential derivatives by formula (11) with $k = .5(-4)$, introduces an error of the order of 10^{-6}. The use of formulas (15) and (16) had a less consistent effect on the errors because the location of the points where the maximum error occurs was observed to change in this case.

TABLE II

First Biharmonic Problem, Maximum Absolute Errors in u

Problem	h	Gupta, Manohar	h	(9)	(10)	(10), (11)	(10), (15), (16)
1	1/5	.6990(−3)	1/4	.2916(−4)	EXACT	.9388(−6)	.1974(−6)
	1/10	.8774(−4)	1/8	.7645(−5)	EXACT	.9835(−6)	EXACT
	1/20	.4768(−5)	1/16	*	*	*	*
2	1/20	.1994(−4)	1/4	.3873(−3)	.2555(−4)	.2555(−4)	.2566(−4)
			1/8	.9492(−4)	.6523(−6)	.4577(−6)	.3667(−6)
			1/16	.3035(−5)	*	*	*
3	1/20	.1981(−1)	1/4	.4372	.4080(−1)	.4080(−1)	.4106(−1)
			1/8	.1035	.2618(−2)	.2618(−2)	.2635(−2)
			1/16	.2594(−1)	.1458(−3)	.1458(−3)	.1259(−3)

However, the order of the method remains the same. Numerically, we estimate the order of the methods by

$$\ln(e_{h1}/e_{h2})/\ln(h1/h2)$$

where e_{h1} and e_{h2} are the maximum errors for two grid mesh widths $h1$ and $h2$, respectively. All the methods based on formulas (10) exhibit order four behaviour.

Since our methods make use of the derivatives u_x and u_y, we give in Table III the

TABLE III

First Biharmonic Problem, Maximum Errors in u, u_x, u_y

	Formulas (9) − $O(h^2)$			Formulas (10) − $O(h^4)$		
			Problem 1			
h	u	u_x	u_y	u	u_x	u_y
1/4	.2916(−4)	.3824(−3)	.9281(−4)	EXACT	.1335(−5)	.1509(−5)
1/8	.7645(−5)	.4076(−4)	.2384(−4)	*	*	*
1/16	*	*	*	*	*	*
			Problem 2			
1/4	.3873(−3)	.1162(−2)	.1162(−2)	.2555(−4)	.1238(−3)	.1238(−3)
1/8	.9492(−4)	.2852(−3)	.2852(−3)	.6523(−6)	.3397(−5)	.3129(−5)
1/16	.3035(−5)	.1009(−4)	.9665(−5)	*	*	*
			Problem 3			
1/4	.4372	.7451	.7451	.4080(−1)	.2468	.2468
1/8	.1035	.3090	.3090	.2618(−2)	.9094(−2)	.9094(−2)
1/16	.2594(−1)	.8307(−1)	.8306(−1)	.1458(−3)	.4621(−3)	.4597(−3)

maximum errors in u, u_x and u_y for the above problems. As expected, the errors for the derivatives are greater than the corresponding errors in u, however, the rate of decrease in the errors in the derivatives is consistent with the order of the methods used.

In Table IV we compare the results for the second biharmonic problem using the formulas which interpolate second derivatives, (12) and (13). As in the case of the first biharmonic problem, in practice we would need to use the boundary formulas (17), (18) in addition to (13). We have not done this in this comparison which is given to indicate the improvement that can be expected with the fourth order method.

6. Concluding Remarks

In this paper we have outlined a procedure for obtaining high order difference formulas for the biharmonic equation. The same procedure can be applied to derive difference formulas for other linear partial differential equations. The procedure described in this paper is different from the one given by Young and Dauwalder [3], Lynch and Rice [4] and Boisvert [5], although all these methods are generalizations of the Mehrstellenverfahren of Collatz [2]. Difference formulas for mesh points near a boundary are obtained without the use of fictitious points, thereby eliminating the usual difficulty encountered in using central-difference methods. A drawback to these methods is that solutions to the resulting linear system of equations cannot be obtained quickly because of the lack of simple structure and positive-definitness. Numerical results for several different model problems have been presented in this paper, and these results along with the work of Lynch, Rice and Boisvert show that high order methods may be attractive at least for those problems where the solution is sufficiently smooth. To make these methods more competitive with the central-difference methods, it is necessary to develop suitable solution techniques that take

TABLE IV

Second Biharmonic Problem, Maximum Absolute Errors in u

Problem	h	Formula (12)	Formula (13)
		$O(h^2)$	$O(h^4)$
1	1/3	.1012(−3)	.1863(−5)
	1/6	.2818(−4)	.1863(−5)
	1/12	.6735(−5)	*
2	1/3	.1458(−2)	.3449(−3)
	1/6	.4326(−3)	.2582(−4)
	1/12	.1043(−3)	.1561(−5)
3	1/3	.3519(1)	.3292(0)
	1/6	.9600(0)	.1979(−1)
	1/12	.2241(0)	.1179(−2)

advantage of the block structure of the resulting linear system of equations. The techniques presented here may open up new avenues of research in the development of discretization schemes for more general problems. The same procedure has now been applied successfully for the Helmholtz equation [11] and the convection-diffusion equation in two and three dimensions [9]. Applications to problems with curved boundaries were reported in Tokyo [10].

Appendix

The special fourth order formulas for the edge and corner cells shown in Fig. 3 are given here in tabular form. In Table V we give the formulas for the first biharmonic

TABLE V

First Biharmonic Problem

	Edge formula (15)			Corner formula (16)		
	$1852u_0 =$	$1852hu_{x0} =$	$43hu_{y0} =$	$95u_0 =$	$41hu_{x0} =$	$41hu_{y0} =$
u_1	426	5892/10	0	22	63/5	12
u_2	506	6	105/4	22	12	63/5
u_3	506	−11052/10	0	26	−123/5	0
u_4	506	6	−105/4	26	0	−123/5
u_5	−667/9	−4346/60	−2/3	−193/36	123/40	123/40
u_6	−43	−1449/10	3	−137/36	191/120	−191/120
u_7	−43	−1449/10	−3	−9/4	−123/40	−123/40
u_8	−667/9	−4346/60	2/3	−137/36	−191/120	191/120
u_9	640/9	1408/3	16/3	32/9	32/3	−32/3
u_{10}	640/9	1408/3	−16/3	32/9	32/3	−32/3
u_{11}				32/9	−32/3	32/3
u_{12}				32/9	−32/3	32/3
hu_{x1}	−589/3	−3698/10	0	−121/12	−163/20	−1/20
hu_{x3}	589/3	−371	0	121/12	−33/4	1/20
hu_{x5}	−41/6	−451/10	−1	0	−41/20	0
hu_{x6}	41/6	−95/2	1			
hu_{x7}	41/6	−95/2	−1	1/3	−1	−21/20
hu_{x8}	−41/6	−451/10	1	−1/3	−21/20	21/20
hu_{y2}	−590/3	−16/10	−35/4	−121/12	−1/20	−163/20
hu_{y4}	590/3	16/10	−35/4	121/12	1/20	−33/4
hu_{y5}				0	0	−41/20
hu_{y6}	−20/3	486/10	−1	− 1/3	21/20	−21/20
hu_{y7}	20/3	−486/10	−1	1/3	−21/20	−1
$h^4f_{0,0}$	35	−1/2	0	43/24	0	0
$h^5f_{1,0}$	0	926/60	0	0	41/120	0
$h^5f_{0,1}$	0	0	1/3	0	0	41/120
$h^6f_{2,0}$	14/3	−1/15	0	17/72	1/120	−1/120
$h^6f_{0,2}$	83/18	−13/30	0	17/72	−1/120	1/120
$h^6f_{1,1}$				0	0	0

TABLE VI

Second Biharmonic Problem

	Edge formula (17)			Corner formula (18)		
	$u_0 =$	$u_{xx0} =$	$u_{yy0} =$	$u_0 =$	$u_{xx0} =$	$u_{yy0} =$
u_1	−.25547873(0)	.12961613(1)	−.31203739(0)	−.22212428(0)	.12676581(1)	−.36593932(0)
u_2	.34592516(0)	−.85136821(0)	.34967075(0)	−.22212428(0)	−.36593932(0)	.12676581(1)
u_3	.34592516(0)	.34863179(0)	−.85032925(0)	.34728955(0)	.34746584(0)	−.85253415(0)
u_4	.34592516(0)	−.85136821(0)	.34967075(0)	.34728955(0)	−.85253415(0)	.34746584(0)
u_5	−.21726104(0)	.44253645(0)	.35893200(0)	−.32705303(0)	.53635963(0)	.53635963(0)
u_6	−.95925163(−1)	.25136821(0)	.25032925(0)	−.21217129(0)	.35070680(0)	.43818699(0)
u_7	−.95925163(−1)	.25136821(0)	.25032925(0)	−.97289555(−1)	.25253415(0)	.25253415(0)
u_8	−.21726104(0)	.44253645(0)	.35893200(0)	−.21217129(0)	.43818699(0)	.35070680(0)
u_9	.42203782(0)	−.66493300(0)	−.37774868(0)	.39958866(0)	−.34147006(0)	−.64574898(0)
u_{10}	.42203782(0)	−.66493300(0)	−.37774868(0)	.39958866(0)	−.34147006(0)	−.64574898(0)
u_{11}				.39958866(0)	−.64574898(0)	−.34147006(0)
u_{12}				.39958866(0)	−.64574898(0)	−.34147006(0)
h^2u_{xx1}	−.60393302(−1)	.60092600(−1)	.14793791(0)	−.61370648(−1)	.60927795(−1)	.14951733(0)
h^2u_{xx3}	−.60393302(−1)	.60092600(−1)	.14793791(0)	−.61370648(−1)	.60927795(−1)	.14951733(0)
h^2u_{xx5}	−.83220582(−2)	.71116476(−2)	.13448732(−1)	0	0	0
h^2u_{xx6}	−.83220582(−2)	.71116476(−2)	.13448732(−1)			
h^2u_{xx7}	−.83220582(−2)	.71116476(−2)	.13448732(−1)	−.74922874(−2)	.64025636(−2)	.12107793(−1)
h^2u_{xx8}	−.83220582(−2)	.71116476(−2)	.13448732(−1)	−.74922874(−2)	.64025636(−2)	.12107793(−1)
h^2u_{yy2}	−.61211000(−1)	.14938091(0)	.60669799(−1)	−.61370648(−1)	.14951733(0)	.60927795(−1)
h^2u_{yy4}	−.61211000(−1)	.14938091(0)	.60669799(−1)	−.61370648(−1)	.14951733(0)	.60927795(−1)
h^2u_{yy5}				0	0	0
h^2u_{yy6}	−.79132091(−2)	.12467494(−1)	.70827877(−2)	−.74922874(−2)	.12107793(−1)	.64025636(−2)
h^2u_{yy7}	−.7913209−(−2)	.12467494(−1)	.70827877(−2)	−.74922874(−2)	.12107793(−1)	.64025636(−2)
$h_2f_{0,0}$	.64197849(−1)	−.14526325(0)	−.14569614(0)	.63629352(−1)	−.14477744(0)	−.14477744(0)
$h^5f_{1,0}$	0	0	0	0	0	0
$h^5f_{0,1}$	0	0	0	0	0	0
$h^2f_{2,0}$	.11983598(−1)	−.17115806(−1)	−.27196614(−1)	.11232310(−1)	−.16473789(−1)	−.25982506(−1)
$h^6f_{0,2}$	.11983598(−1)	−.26042216(−1)	−.16586707(−1)	.11232310(−1)	−.25982506(−1)	−.16473789(−1)
$h^6f_{1,1}$				0	0	0

problem. In Table VI we give the corresponding formulas for the second biharmonic problem. A blank in the table means that the quantity is not used in that particular formula.

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